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Finite Boolean Algebras and Subgroup Lattices Of Finite Abelian (

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Abstract

We use the fundamental theorem of finite abelian groups to give a characterization of when the su
a finite abelian group can be given the structure of a Boolean algebra. Arguments involving both c
in a Boolean algebra as well as the distributivity of a lattice are used. Prerequisites include an elen
in abstract algebra as well as familiarity with elementary lattice theory.

Introduction

We shall follow the notation in [4] and suggest this as a reference for unexplained terminology as

We consider only finite abelian groups in this paper. We use the notation Z_n for the cyclic group of
modulo n. The direct product of groups G and H is denoted by G ´ H. It is a fact that every finite a
isomorphic to a direct product of cyclic groups of prime power order. This result is important to ou
state it below as Theorem 1. For a proof, we refer the reader to [4], pages 200-203.

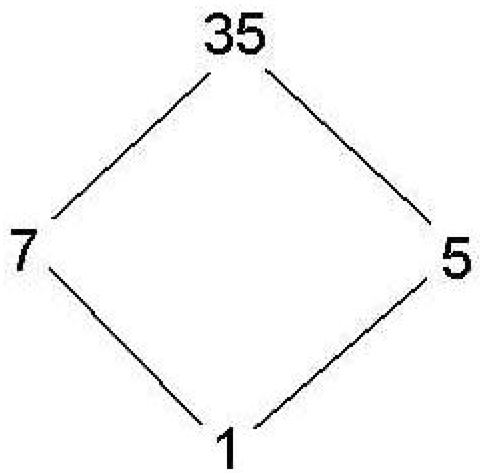
Theorem 1. *Every finite abelian group G is isomorphic to a direct product of cyclic groups of the f*

$$Z_{p(1)} ´ Z_{p(2)} ´ Z_{p(3)} ´ \ldots ´ Z_{p(n)}$$

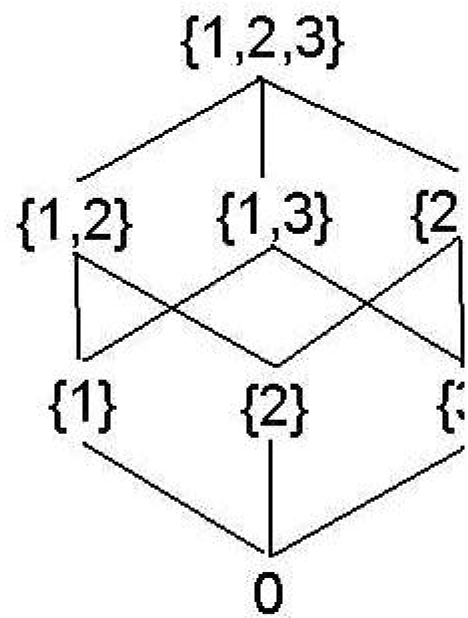
where the p(i)'s are powers of primes and the primes are not necessarily distinct.

Boolean Algebras and Subgroup Lattices

A Boolean algebra is a lattice that is both distributive and complemented. Figure 1 shows the Hass
two finite Boolean algebras.



The divisors of 35



The set of all subsets of a three

Figure 1

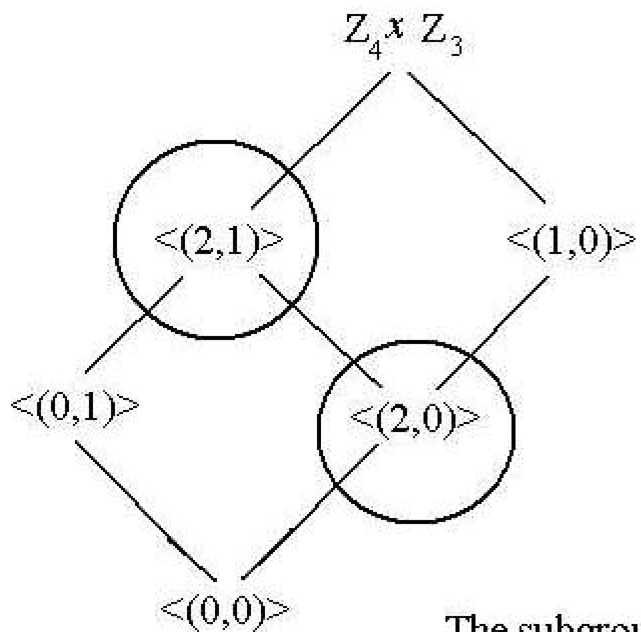
For the lattice of divisors of an integer, the partial order is given by a $\leq$ b if a divides b. It is easy the supremum of two elements a and b is given by the least common multiple of a and b. Similarly of a and b is just the greatest common divisor of a and b. The lattice of divisors of an integer n has structure as the lattice of all subgroups of Z_n. If H and G are subgroups of a given group, then the the subgroup lattice is given by H $\leq$ G if H is a subgroup of G. The lattice operations are then as fo infimum of H and G is simply H∩G and the supremum of H and G is the smallest subgroup that co and G (the subgroup generated by H and G). We denote the supremum of H and G by H∨G.

The divisor lattice of an integer n can be given the structure of a Boolean algebra if and only if n is (e.g. [1] page 182, number 7 or [4] page 322, number 2). One can immediately infer that the latt of Z_n can be given the structure of a Boolean algebra if and only if n is square free. For the remain paper, we shall demonstrate that the only finite abelian groups with Boolean subgroup lattices are where n is square free. We acknowledge that the latter result is a straightforward application of Th we were unable to find this result in the literature. We draw inspiration from the following example

Example. Let G = Z4´ Z3. This group is cyclic (see [3] Theorem 8.2) and therefore every subgrou described by a generator. We write = H if the element h generates H. The subgroups of G are

{G, <(1,0)>, <(0,1)>, <(2,1)>, <(2,0)>, <(0,0)>}.

The Hasse diagram for the subgroup lattice of G is given in Figure 2.



The subgroup lattice of $Z_4 x Z_3$

The circled elements have no complement.

Figure 2

Note that we can find elements in the lattice which do not have complements and therefore the lat
Boolean. We point out that one can also use the fact that the number of elements in the lattice is
(see [2] Chapter 3, Theorem 4). Inspired by the latter example, we state and prove the following

Lemma 2. *Suppose that G = $Z_{p(1)}$ ´ $Z_{p(2)}$ ´ $Z_{p(3)}$´ ...´ $Z_{p(n)}$ where each p(i) is a power of a prime. (i) is not prime, then the subgroup lattice of G is not Boolean.*

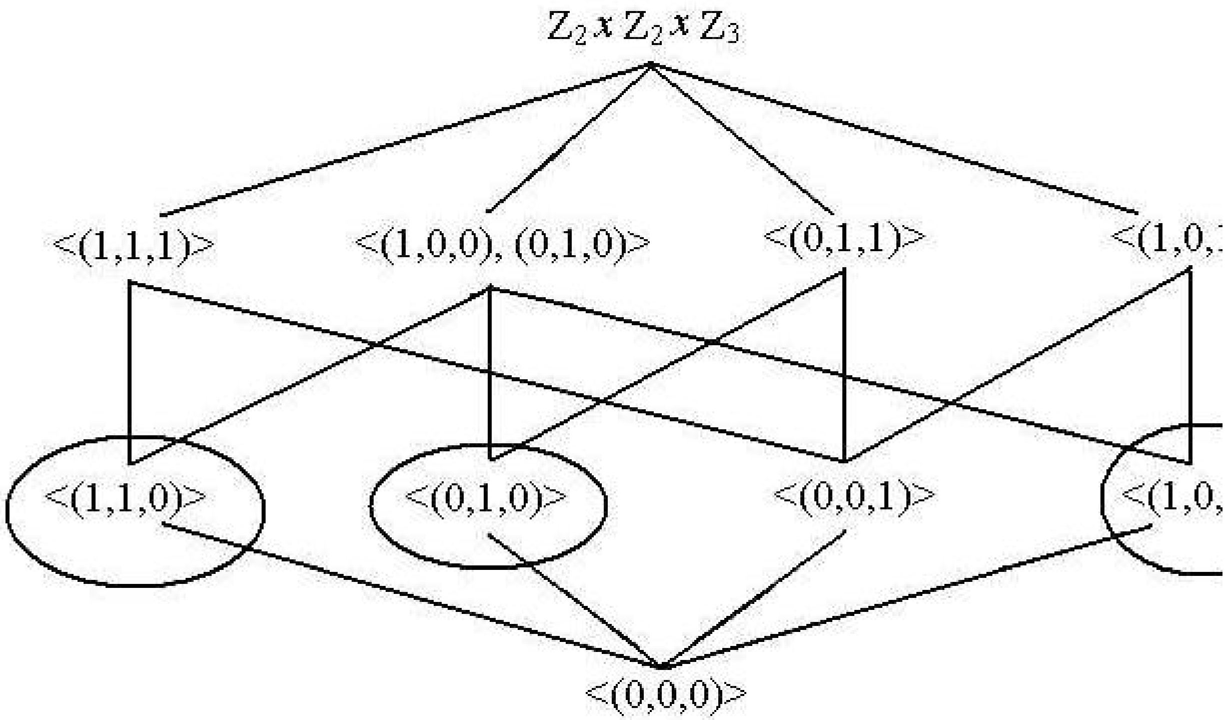
Proof. The case where there is only one summand in the product is easy and left for the reader. Su is not a prime number and is a power of the prime number q. Let H denote the subgroup generate element with a q in the ith coordinate and zeros in all other coordinates. Let K be any subgroup su {(0,0,0,...,0)}. The supremum of H and K can never be all of G since the subgroup generated by H have all elements of $Z_{p(i)}$ in the i^{th} coordinate. It follows that the subgroup lattice of G cannot be B

This does not take care of all possible cases in the class of finite abelian groups. We need another

Example. Let G = Z_2´ Z_2´ Z_3. In the event that there are repeated primes, the group G is not cyc Theorem 8.2 in [3]). One has to be careful about listing the subgroups. There are 8 cyclic subgrou

{<(0,0,0)>, <(1,0,0)>, <(0,1,0)>, <(0,0,1)>, <(1,1,0)>, <(1,0,1)>, <(0,1,1)>, <(1,1

The remaining subgroups which are not cyclic are the whole group and <(1,0,0), (0,1,0)>. The Ha
the subgroup lattice of G is given in Figure 3.



The subgroup lattice of $Z_2 x Z_2 x Z_3$

The circled elements violate the distributive law.

Figure 3

Note that we can find elements in the lattice which violate the distributive law. Again, the example
following lemma.

Lemma 3. *Suppose that G =* $Z_{p(1)} \times Z_{p(2)} \times Z_{p(3)} \times \ldots \times Z_{p(n)}$ *where each p(i) is a prime. Suppose fu =p(j) for some i not equal to j. Then the subgroup lattice of G is not Boolean.*

Proof. Without loss of generality, assume that the first two groups in the product are the same. Le

H = <(1,0,0,0,0,...)>, K = <(0,1,0,0,0,...)> and L = <(1,1,0,0,0,...)>.

We have that $H \vee (K \cap L) = H$. However, $(H \vee K) \cap (H \vee L) = \langle H,K \rangle$. The latter argument can be visuali the three circled elements on Figure 3 as H, K and L in that order. It follows that the subgroup latt distributive and therefore not Boolean.

Combining Lemma 2 with Lemma 3 yields Theorem 4, which is our main result.

Theorem 4. If G is a finite abelian group, then G has a subgroup lattice which is a Boolean algebr is isomorphic to Z_n where n is a square free integer.

Proof. By Theorem 1, $G = Z_{p(1)} ´ Z_{p(2)} ´ Z_{p(3)} ´ \ldots ´ Z_{p(n)}$ where p(i) is a power of a prime for every all powers are 1. By Lemma 3, there are no repeated primes. We now apply Corollary 8.12 in [4] t Z_n where n = p(1)´ p(2)´ ... ´ p(n). This is, of course, a square free integer.

Extensions

It is known that the normal subgroups of any group form a modular lattice (see Theorem 11 in [2] modular, then it is obviously distributive. Thus, when studying the structure of the subgroup lattic groups, one may first consider the lattice of normal subgroups. Perhaps a place to start would be permutation groups since by Cayley's Theorem, every finite group can be represented as a subgro permutation group.

References

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