Variational Methods in $\lambda\phi^4$ Quantum Field Theory

by

KYLE S. CRANMER

Advisor: Dr. Paul Stevenson

A dissertation submitted in partial fulfillment of

the requirements for the degree of

Bachelor of Arts in Physics at

at

Rice University

Houston, Texas

Spring 1999

Variational Methods in $\lambda\phi^4$ Quantum Field Theory

Abstract

The dynamics of $\lambda\phi^4$ scalar quantum field theory is studied using two different variational approaches: the Gaussian effective potential and a BCS-motivated trial state. Simultaneous minimization of both of the BCS variational parameters yields a "reduction in generality", in which the BCS-generalization yields the Gaussian case. "End-point" contributions to the variational approach produce instability when the coupling constant is negative and infinitesimal but no effect when the coupling constant is positive and infinitesimal. The dynamics of the negative, infinitesimal case are presented.

Acknowledgments

I would like to thank my research advisor Professor Paul M. Stevenson not only for his guidance in the research presented in this thesis, but also the guidance he has provided me in becoming a mature physicist. My Sophomore year it was my privilege to learn the beautiful theory of classical mechanics under his mathematically rigorous yet extraordinarily intuitive instruction. My Junior year it was my pleasure to be introduced to relativistic quantum mechanics in an equally enlightening and sleep-depriving manner. This, my Senior, year he once again instructed me in scalar quantum field theory. When I look at the world I see physics and when I see physics I see it with eyes primarily influenced by this man. His influence will be ever-present and ever-lasting in my carrer on this planet.

I am utterly grateful to my parents, Joan and Morris, for their support throughout my college experience and their fostering throughout my childhood. I don't know what they did right, but I am grateful for the well rounded, stable, and inspired life they sculpted. Furthermore, I am forever grateful for my mother's encouragement for me to be observant and appreciative of nature and my father's effort and patience in answering all of my "why is the sky blue" questions.

For my brother, Dylan, I hold a special place in my heart. Since I was young I have felt that we have very similar views about life and the world. Interestingly, he has chosen to express his thoughts and feelings through music where I have chosen physics. It has always amazed me how similar our motivations are for doing what we do despite the apparent contrast. Indeed his impact has helped me appreciate physics, music, and the gulf in between.

I would like to thank Natasha Lavelle not for her guidance or support, but for the inspiration which she has instilled in me. Physics is after all about understanding nature. The appreciation of the symmetry of a flower and the complexity of a tree are every bit as important to me as the symmetries of the Standard Model and the complexities of Quantum Field Theory. The impact she has had on my view of the world and the enthusiasm with which I study it would be difficult to overstate.

To Cynthia Duggan I feel deeply indebted. Both her willingness and enthusiasm to listen to me ramble on about what I have learned and what I am thinking about have given me great confidence and enriched my life.

I would like to thank J Provine for painstakingly working his way through a very rough draft of this thesis and the advice he offered.

Without the excellent instruction from the Rice University physics department and my two internships at Bonner Laboratory I would not be where I am today. Specifically, Hannu E. Miettinen's introduction to particle physics in our multivariate search for new physics at DØ has served to be one of my greatest assets as a physicist and for that I am grateful. Furthermore, the instruction of Billy Bonner, Jim Hannon, Huey Huang, Randy Hulet, F. Curtis Michel, and Jon Weisheit have all placed seeds of understanding, unique to their vision, which are now incorporated into my own. Lastly, I would like to thank Marjorie Corcoran, Geary Eppley, Paul Padely, and Pablo Yepes for their friendship and covert influence in my life.

Contents

List of Figures

1 Quantum Field Theory and Variational Methods

1.1 Introduction

It is a unique experience to present this thesis in the twilight of the 20^{th} century. The last century has seen the advent of the two most successful theories to ever be formulated - perhaps two of the most profound accomplishments of our civilization: General Relativity and the Standard Model. General Relativity, formulated by Albert Einstein, is a theory of the very large, it describes the structure of space-time better known as gravity. The Standard Model is a theory of the very small, it describes the fundamental particles and their interactions with the strong and electroweak forces. The Standard Model has been tested to an unbelievable accuracy; however, it relies on the existance of a particle which has never been observed: the Higgs boson.

The language of the Standard Model is Quantum Field Theory (QFT). A scalar quantum field is the simplest quantum field that can be studied. The simplest theory beyond that of a free field theory (no interactions) is that of ϕ^4 QFT (where ϕ^4 refers to the interaction term in the Lagrangian), and in fact it is a ϕ^4 scalar field theory which describes the Higgs Boson.

Unfortunately, for any theory more complicated than a free field theory, the dynamics of the field are too complicated to be studied exactly. In order to make progress some approximation method must be used. Two major tools theorists use are perturbation theory and "effective" potentials. Perturbation theory essentially relies on expanding central quantities with respect to quantum corrections and studying only leading order terms. Perturbation theory is perhaps the most powerful tool in physics today; however, it has the drawback that it breaks down as the quantum corrections become large. Another approach is to formulate an "effective" potential that inherently compensates for quantum effects and study that potential in a classical manor. In fact, there are several "effective" potentials formulated to simplify QFTs, all with different properties, advantages, and disadvantages. One attractive option, and of particular interest to the work presented here, is that of the Gaussian effective potential, which uses the expectation value of the "ground" state energy of a particle as the effective potential. Of course, knowing the ground state is in itself a task, so this must be an ansatz. The purpose of this thesis is to try to understand the relationship between the conclusions made by studying ϕ^4 QFT with similar effective potentials and two different ansatz for the ground state of that field.

1.2 Quantum Field Theory

Quantum Field Theory (QFT) is a theory which describes elementary particle physics. QFT came about in attempt to find a formalism that could describe a many-particle universe in a Lorentz invariant framework. A brief introduction to the relevant portions of scalar field theory will be given here.

The fundamental object studied in QFT is the field, denoted by ϕ. ϕ is both a function of both position and time. The physics results from studying a quantity known as the *action*, $S[\phi]$, which is a functional of the field. The action is defined by:

$$S[\phi] = \int dtL = \int dt \int d^3x\mathcal{L} = \int d^4x\mathcal{L}, \tag{1.1}$$

where

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V[\phi]. \tag{1.2}$$

$\mathcal{L}$, the Lagrangian density, is expressed in a form which is a product of covariant and contravariant terms - this insures that the theory is "manifestly covariant". The Functional Integral Formalism, Feynman's Path Integral Formalism, is carried out fully with the Lagrangian as the central quantity; however, it can be useful to introduce the Hamiltonian formalism. The definition is an abstraction of the familiar classical mechanics formalism. The general coordinate "q_j" is replaced by the field - which is no longer labeled by a discrete index j, but by a continuous label $\vec{x}$. "q_j" $\longrightarrow \phi_{\vec{x}}(t) = \phi(\vec{x}, t)$. Thus, we define the Hamiltonian:

$$H \equiv \sum_j p_j\dot{q}_j - L \longrightarrow \int d^3x\mathcal{H} = \int d^3x\dot{\phi}_x\dot{\phi}_x - \int d^3x\mathcal{L}. \tag{1.3}$$

Note that in the definition of the Hamiltonian, we have implicitly picked a reference frame.

In a free field theory, the particles of the field do not interact. In this case $V[\phi] = \frac{1}{2}m^2\phi^2$ (*i.e.* a simple harmonic oscillator at every point). The general solution to the free field is a superposition of normal modes. Leaving out some normalization issues we find:

$$\phi(\vec{x}, t) = \int(dk)[a(\vec{k})e^{-ik.x} + a^\dagger(\vec{k})e^{ik.x}], \tag{1.4}$$

where $k.x \equiv k^\mu x_\mu = \omega_k t - \vec{k}\cdot\vec{x}$, $w_k \equiv \sqrt{k^2 + m^2}$, and $(dk) \equiv \frac{d^3k}{2(2\pi)^3\omega_{\vec{k}}}$. The main point here is that now the field is expressed in terms of the familiar raising and lowering (creation and annihilation) operators and that particles can be thought of as particular oscillations in the field. The vacuum state, denoted by $|0\rangle$, is the state in which all of the SHO's are in there ground state (*i.e.* no particles).

1.3 $\lambda\phi^4$ Scalar Quantum Field Theory

So what is meant by $\lambda\phi^4$ QFT? The $\lambda\phi^4$ refers to an additional term in the potential energy of the field: an interaction. In terms of the particles, the interaction corresponds to a hard-core scattering, where the point-like particles bounce off each other.

The Lagrangian density associated with $\lambda\phi^4$ QFT is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_B^2\phi^2 - \lambda_B\phi^4, \tag{1.5}$$

which leads to a Hamiltonian density

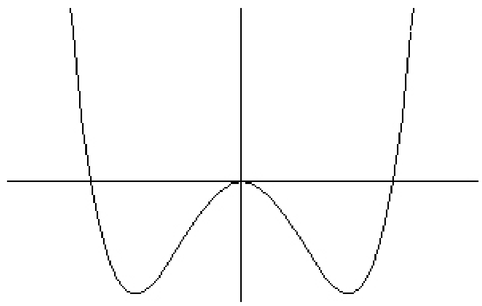

Figure 1: The double-well potential.

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_B^2\phi^2 + \lambda_B\phi^4. \tag{1.6}$$

The Lagrangian expressed in (1.5) leads to two distinct physical situations. The first situation corresponds to $m_B^2 > 0$ - the familiar anharmonic oscillator. The second situation, $m_B^2 < 0$, results in a double-well potential (see Figure 1). This double-well potential may lead to a vacuum state which does not respect the symmetry of the Lagrangian. The symmetry in the Lagrangian is said to be *spontaneously broken* when "Nature picks" which side of the well to fall into. By doing so, the vacuum expectation of the field becomes non-zero ($\langle\phi\rangle = \phi_0$).

The motivation for studying $\lambda\phi^4$ is related to the Higgs' Mechanism and the origin of the masses of elementary particles. The key point is that if there exists a scalar field (the Higgs field) which couples to the fields associated with other elementary particles and that field has a non-zero expectation value, then the coupling would produce an "effective mass" term. It is the $\lambda\phi^4$ term which gives rise to a double-well potential, thus giving rise to Spontaneous Symmetry Breaking (SSB) and ultimately to the origin of the masses of all elementary particles.

1.4 Effective Potentials

Before discussing the concept of an effective potential in a field-theoretic sense, let us first consider the use of an effective potential in classical problem: central-force motion. The dynamics of two particles of mass m_1 and m_2 attracted by a potential function $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ can be reduced to an equivalent one-body problem of a particle of reduced mass $\mu \equiv m_1m_2/(m_1 + m_2)$ in an effective potential

$$V_{\text{eff}}(r) \equiv V(r) + \frac{l^2}{2\mu r^2}. \tag{1.7}$$

"V_{eff} is a fictitious potential that combines the real potential function $V(r)$ with the energy term associated with the angular motion about the center of force [5]." The idea is that it is the particle *sees* the effective potential - the complications arising from the angular motion of the particles is inherently taken into account by the effective potential. Figure 2

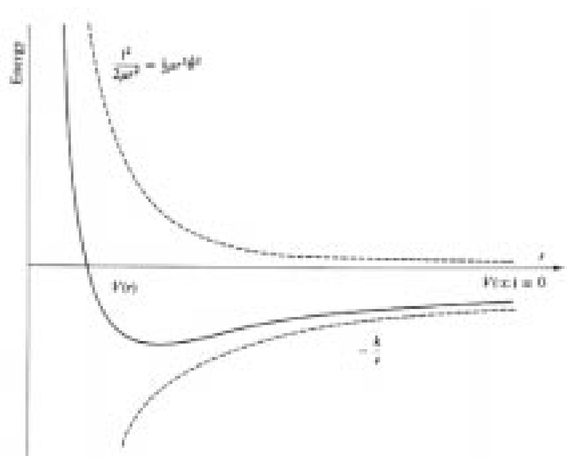

Figure 2: Marion and Thornton's illustration of the effective potential arising from a $1/r$ potential with non-zero angular momentum.

shows how the angular motion contributions to the effective potential change the nature of a inverse-square-law force. Conservation of angular momentum results in an effective barrier restricting the motion of the two particles. This example is a strong motivation for the use of an effective potential. Prior to moving to the equivalent one-body problem, it would be difficult, mathematically, to see what would keep the two particles from spiraling into each other.

In quantum mechanics the notion of an effective potential can be extended. Due to quantum fluctuations inherent in all quantum mechanical systems, even the most tame potentials can give rise to counter-intuitive results. The most well known of case of quantum fluctuations producing dramatically different physical consequences from the classical counterpart is the Hydrogen atom - a quantum mechanical two-body problem. In addition to the repulsion of the electron from the centripetal barrier found in the classical case (1.7), there exists an "effective repulsion" from the nucleus even in the zero angular momentum s-state. This "repulsion" is due to the uncertainty principle. Despite the initially surprising results we find in quantum mechanical systems, we can partially summarize our quantum mechanical intuition with the mantra *a quantum mechanical particle does not like to live in a narrow potential well*. It is this "quantum claustrophobia" which prevents the electron from spiraling into the nucleus and our universe from disintegrating. Figure 3 illustrates the shape of an effective potential that inherently takes into account the quantum fluctuations. Again, the goal is to find an effective potential to which we can apply our classical intuition.

In Section 2 we will formally present the Gaussian effective potential; however, a more conceptual description of the properties we want in a good effective potential may first be useful. Figure 4 shows four related potentials which each yield unique physics. In the first case (Figure 4a), the quantum effects are small. Because the quantum effects are small the corrections to the exact potential are minimal and the double-well shape of the effective potential demonstrates a very real barrier. The most important features of the effective

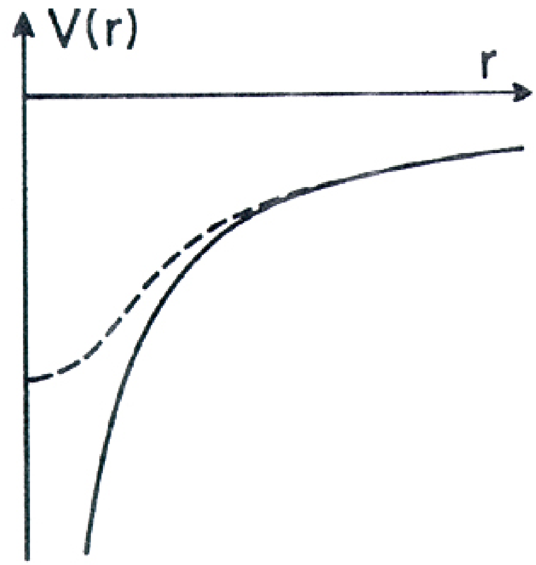

Figure 3: Stevenson's sketch of the effective potential for a s-state of Hydrogen.

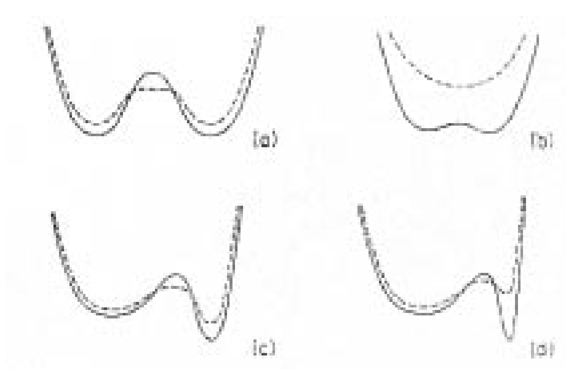

Figure 4: Stevenson's "Effective" potentials for some double-well examples.

potential are that it is slightly higher in the wells due to the zero-point energy $\frac{1}{2}\hbar\omega_{\rm eff}$ and lower in barrier region because the quantum mechanical particle is able to spread out into the wells. As the quantum effects become large (Figure 4b) the "barrier" becomes no more than an inconsequential bump in an essentially single-well potential. Now consider the more illustrative example of an asymmetric potential with moderate quantum effects (Figure 4c). Again we see that the effective potential is higher in the wells and lower in the barrier region. Note that in the narrower well the effective potential diverges from the exact potential more than in the wider, shallower well. This is because $\omega_{\rm eff}^2 \equiv \frac{1}{m}d^2V(x)/dx^2$ is larger in the narrower well. In fact, as Figure 4d illustrates, for a sufficiently narrow well a quantum mechanical particle will prefer to sit in the broader well even though it has a higher classical energy. This example clearly demonstrates the utility of a well-behaved effective potential. The next task is to attempt to define such a quantity.

In the case of QFT, our need for an "effective" potential is even greater because our intuition is even less developed than in the case of quantum mechanics. There are several "effective" potentials that have been proposed: "the effective potential", the one-loop effective potential, the Gaussian effective potential, and the BCS-generalization of the Gaussian effective potential. These all have their own advantages and disadvantages. For instance, "the effective potential" defined by

$$V_{\rm eff}(\phi_0) = \min_{\psi}\langle\psi|H|\psi\rangle, \tag{1.8}$$

with ψ subject to

$$\langle\psi|\psi\rangle = 1 \qquad \textit{and} \qquad \langle\psi|\phi|\psi\rangle = \phi_0, \tag{1.9}$$

has the important property that its global minimum gives the exact ground state energy of the system [1]. However, it also has the undesirable property that it is convex (*i.e.* $d^2V_{\rm eff}/d\phi_0^2 \geq 0$) [9, 10] precluding it from ever having a double-well shape. This alone excludes "the effective potential" from being the formulation of what we had in mind in Figure 4. An order-$\hbar$ approximation of $V_{\rm eff}$ known as the one-loop effective potential (1LEP) has also become a popular tool for field theorists. It is interesting that this method has gained so much popularity given that it generally breaks down when quantum effects become large - precisely the area of interest. A superior quantity, the Gaussian effective potential (GEP) presented formally in Section 2 produces more reliable results both qualitatively and quantitatively. [1] Unfortunately, the GEP does not enjoy the property that the global minimum of $\bar{V}_G$ gives the exact ground state energy, as in the case of "the effective potential". The Rayleigh-Ritz theorem, however, insures that $\bar{V}_G \geq V_{\rm eff}(\phi_0)$ at any ϕ_0. Furthermore, as will be emphasized in the next section, one can expect a good approximation to the ground state energy for any respectable guess of the ground state.

[1] Comparisons of various "effective" potentials are made in the case of quantum mechanics where we have better intuitive understanding and exact results are often possible. How the different quantities generalize to QFT is a matter of conjecture.

1.5 Variational Approaches

The variational method of quantum mechanics is a useful computational tool as well as an excellent way to foster trust and intuition for quantum mechanics. For any state ψ we have

$$|\psi\rangle = \sum_n c_n |E_n\rangle, \tag{1.10}$$

where $|E_n\rangle$ are the exact eigenstates of a system (with corresponding eigenvalues E_n). Thus we have

$$\langle\psi|H|\psi\rangle = \sum_n |c_n|^2 E_n \geq \sum_n |c_n|^2 E_0 = E_0. \tag{1.11}$$

Thus for any state $|\psi\rangle$ the expectation value of the energy is higher than the ground state energy. This motivates the key idea to variational approaches: by assuming a parametrized set of trial states $|\psi(\alpha_1, \alpha_2, \alpha_3, ...)\rangle$, and minimizing $\langle\psi|H|\psi\rangle$ with respect to those parameters, we can make a good estimate of the ground state and the ground state energy.

An instructive example is that of the Helium atom. By assuming the trial state $|1,0,0\rangle_{\tilde{Z}}$, the single-particle ground state of a hydrogenic atom parametrized by nuclear charge $\tilde{Z}$, for both electrons in the Helium atom one obtains upon minimization that $\tilde{Z} = Z - \frac{5}{16}$. This yields to an energy expectation value of -77.4 eV compared to an experimental value of $E_0 =$ -79.0 eV - an improvement on the first-order perturbative result using the electron-electron repulsion as a perturbation. In fact, Perkeris achieved an estimate of the ground-state energy of Helium that agreed within experimental errors by using a trial state involving 1075 parameters [6].

In Section 1.4 "the effective potential" was defined by Equation 1.8. The minimization of the energy expectation value with respect to the variational parameter (in this case a function) ψ makes V_{eff} a variational tool. Unlike the previous example, this is a functional minimization. For the Gaussian effective potential the minimization is again with respect to variable. In Section 3 a generalization of the Gaussian effective potential will be introduced. For the BCS-generalization, the minimization occurs for two parameters, a function and an ordinary variable.

There are two important points to be made about variational methods, one logical and one technical. Though one may work long and hard to find a trial state from the optimization of a parametrized set of trial states, that does not mean that a trial state picked out of the blue is not worth being studied. In fact, if by ansatz one discovered a state which had a lower energy expectation value than their optimized trial state, then the ansatz is now the best approximation of the ground-state available. Furthermore, it is the *global* minimum of the parametrized set of trial states which one seeks to find. If the parameter space is not open, this does not necessarily correspond to $\partial\langle E\rangle/\partial\alpha_i = 0$; one may need to investigate $\langle E\rangle$ at the end points of the parameter range..

Up until this point, much of what has been said has been intentionally vague. For instance, most of the description of "effective" potentials and variational methods have varied from quantum mechanical language to QFT language. In Section 2.1 the relationship between quantum mechanics and QFT will be elucidated.

2 The Gaussian Effective Potential

The Gaussian effective potential (GEP) is an elegant nonperturbative tool for studying quantum field theory [1, 2]. Essentially the GEP concept is a formulation of our intuitive understanding of quantum mechanical fluctuations that can be generalized to QFT. The concept has its own rich history having been reinvented several times. Compared to other field-theoretic "effective" potentials, such as the one-loop effective potential (1LEP), the Gaussian effective potential has been advocated by Stevenson [1] as being (i) conceptually superior, (ii) more reliable, in both quantitative and qualitative terms, and (iii) almost equally easy to calculate.

2.1 A Quantum Mechanical Analogy

The Gaussian effective potential has a very clear interpretation when applied to quantum mechanics. Indeed, it is in its application to quantum mechanics that its effectiveness as a variational method has been established. By constructing a clear analogy with QFT (in 0-spatial and 1-temporal dimensions) the Gaussian effective potential can be generalized to any dimensionality.

Consider the 1-dimensional quantum mechanical system governed by the following Hamiltonian:

$$H = \frac{1}{2}p^2 + V(\phi), \tag{2.1}$$

where ϕ and p represent the position and momentum of a particle with unit mass, respectively. By specifying a $V(\phi)$ (*i.e.* a potential energy associated with every spatial position ϕ) the dynamics of the system can be calculated using Schrödinger's Equation. The analogue to this system in QFT is a quantum field, ϕ, restricted to a spacetime consisting of (one point)$\otimes$(time) - "(0+1)-dimensions" - governed by the Lagrangian

$$L = \int dt[\frac{1}{2}(\partial_t\phi)(\partial_t\phi) - V(\phi)], \tag{2.2}$$

essentially (2.1) with $p = -i\partial_t\phi$. Now, $V(\phi)$ is the potential associated with the field (it can no longer be a function of position at all because there is only one location available in this spacetime). When $V(\phi) = \frac{1}{2}m^2\phi^2$ (2.2) corresponds to a free field theory.

In order to generalize to (3+1)-dimensional QFT, it may seem natural to generalize (2.1) by sending the spatial coordinate $\phi \to \vec{\phi} = (\phi_x, \phi_y, \phi_z)$ resulting in

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_z^2 + V(\phi_x, \phi_y, \phi_z) \tag{2.3}$$

However, the QFT analogue to (2.3) is *not* a (3+1)-dimensional quantum field, $\phi = \phi(\vec{x}, t)$, but instead a (0+1)-dimensional field theory with three fields, $\phi_x(t)$, $\phi_y(t)$, and $\phi_z(t)$ coupled with interactions described by $V(\phi_x, \phi_y, \phi_z)$.

The primary reason for "the effective potential" (1.8) failing to realize our ideal "effective" potential (see Figure 4) is the possibility that the trial wave function may not truly be located at ϕ_0. [2] By constructing a quantity that insures that the wave function is concentrated in the vicinity of ϕ_0 we can hope to, and indeed do, achieve more physically significant results from our effective potential. An obvious and natural way of constructing our trial state is to place a Gaussian wave function centered on ϕ_0. By parametrizing our trial states with Ω, the width of the Gaussian, we have constructed an approximation to "the effective potential". Thus we define the GEP as follows:

$$\bar{V}_G(\phi_0) \equiv \min_{\Omega} V_G(\phi_0, \Omega) \equiv \min_{\Omega}{}_{\phi_0,\Omega}\langle\psi|H|\psi\rangle_{\Omega,\phi_0}, \tag{2.4}$$

with

$$\langle\phi|\psi\rangle_{\Omega,\phi_0} = \left(\frac{\Omega}{\hbar\pi}\right)^{1/4} \exp\left[-\frac{1}{2}\frac{\Omega}{\hbar}(\phi-\phi_0)^2\right]. \tag{2.5}$$

It is important to be clear that the parameter Ω is obtained by minimizing $\langle H\rangle$ at each ϕ_0, so Ω is itself a function of ϕ_0. It would be selling the GEP short to think of it as a mere approximation of V_{eff}. In fact, $\bar{V}_G$ has several advantages to V_{eff} not the least of which being that it need not be convex. Unfortunately, we have lost the property that the global minimum of $\bar{V}_G$ gives the ground-state energy. Furthermore, it is important to note that the GEP is still readily calculable.

In order to generalize the GEP to QFT we make use of the creation-annihilation operator formalism. This is achieved with the substitutions

$$\phi = \phi_0 + \hbar(2\hbar\Omega)^{-1/2}(a_\Omega + a_\Omega^\dagger), \tag{2.6}$$

$$p = \frac{1}{2}i(2\hbar\Omega)^{1/2}(a_\Omega - a_\Omega^\dagger), \tag{2.7}$$

where

$$[a_\Omega, a_\Omega^\dagger] = 1 \tag{2.8}$$

and the evaluation of $\langle H\rangle$ in the state $|0\rangle_{\Omega,\phi_0}$, defined by

$$a_\Omega|0\rangle_{\Omega,\phi_0} = 0. \tag{2.9}$$

In the quantum mechanical case, $|0\rangle_{\Omega,\phi_0}$ is the ground-state of the simple harmonic oscillator (SHO) with frequency Ω centered at ϕ_0. In field-theoretic terms $|0\rangle_{\Omega,\phi_0}$ corresponds to the vacuum state with mass Ω and the field shifted by ϕ_0. Now with the use of (1.4) and (1.3) it is easy to see how to express the field theoretic generalization to the GEP:

$$\bar{V}_G(\phi_0) \equiv \min_{\Omega} V_G(\phi_0, \Omega) \equiv \min_{\Omega}{}_{\phi_0,\Omega}\langle 0|H|0\rangle_{\Omega,\phi_0}. \tag{2.10}$$

[2] The condition given in (1.9) insures only that the expectation value of the wave function is ϕ_0, which does not imply that there is any amplitude for the particle to be there. In this sense, the trial wave function may be wholely uninformed about the physical situation at ϕ_0 and thus V_{eff} may not be a reliable indication of the physics.

2.2 The GEP for $\lambda\phi^4$ QFT

First we start by writing down the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_B^2\phi^2 - \lambda_B\phi^4 \tag{2.11}$$

which by virtue of

$$\phi = \phi_0 + \int (dk)_\Omega[a_\Omega(\vec{k})e^{-ik.x} + a_\Omega^\dagger(\vec{k})e^{ik.x}] \tag{2.12}$$

and

$$\partial_\mu\phi = \int (dk)_\Omega(-ik_\mu)[a_\Omega(\mathbf{k})e^{-ik.x} - a_\Omega^\dagger(\mathbf{k})e^{ik.x}] \tag{2.13}$$

yields the corresponding Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_B^2\phi^2 + \lambda_B\phi^4, \tag{2.14}$$

where $k.x \equiv k^\mu x_\mu = \omega_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x}$ and

$$k_0 = \omega_\mathbf{k}(\Omega) \equiv (\mathbf{k}^2 + \Omega^2)^{1/2}. \tag{2.15}$$

In ν spatial dimensions the integration measure is

$$(dk) \equiv \frac{d^\nu k}{2(2\pi)^\nu\omega_\mathbf{k}(\Omega)}. \tag{2.16}$$

Evaluating $V_G(\phi_0, \Omega)$ (2.10) is relatively straightforward; we must compute ${}_{\phi_0,\Omega}\langle 0|H|0\rangle_{\Omega,\phi_0}$ in the state $|0\rangle_{\Omega,\phi_0}$ that is annihilated by $a_\Omega(\mathbf{k})$. Separating the calculation term by term we have the kinetic term

$$\langle 0|\tfrac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2]|0\rangle = \int (dk)[\omega^2 - \frac{1}{2}\Omega^2], \tag{2.17}$$

the free field term

$$\langle 0|\tfrac{1}{2}m_B^2\phi^2|0\rangle = \frac{m_B^2}{2}\left[\phi_0^2 + \int (dk)\right], \tag{2.18}$$

and finally the interaction term

$$\langle 0|\lambda_B\phi^4|0\rangle = \lambda_B\left[\phi_0^4 + 6\phi_0^2\int (dk) + 3\int (dk)\int (dk')\right]. \tag{2.19}$$

With the notation

$$I_N(\Omega) \equiv \int (dk)_\Omega[\omega_\mathbf{k}^2(\Omega)]^N \tag{2.20}$$

we can write

$$V_G(\phi_0, \Omega) = I_1 + \frac{1}{2}(m_B^2 - \Omega^2)I_0 + \frac{1}{2}m_B^2\phi_0^2 + \lambda_B\phi_0^4 + 6\lambda_B I_0\phi_0^2 + 3\lambda_B I_0^2. \tag{2.21}$$

In order to obtain the GEP we must minimize $V_G(\phi_0, \Omega)$ with respect to the variational parameter Ω. As noted in Section 1.5 the global minimum does not necessarily correspond to (2.22) because $\Omega \in [0, \infty)$. So, before we are done we will need to examine $V_G(\phi_0, \Omega = 0)$ and $V_G(\phi_0, \Omega \to \infty)$. However, for now let us examine the local minima

$$\left.\frac{dV_G}{d\Omega}\right|_{\Omega=\bar{\Omega}} = 0. \tag{2.22}$$

The optimal value of Ω, denoted by $\bar{\Omega}$, will in general be a function of ϕ_0 given by the "$\bar{\Omega}$ equation" or the "gap equation". Jumping ahead we have the relation from [2]

$$\frac{dI_N}{d\Omega} = (2N-1)\Omega I_{N-1}, \tag{2.23}$$

one obtains an expression for the $\bar{\Omega}$ equation[3]

$$\bar{\Omega}^2 = m_B^2 + 12\lambda_B[I_0(\bar{\Omega}) + \phi_0^2] \tag{2.24}$$

which leads to [4]

$$\bar{V}_G(\phi_0) = I_1 - 3\lambda_B I_0^2 + \frac{1}{2}m_B^2\phi_0^2 + \lambda_B\phi_0^4. \tag{2.25}$$

2.3 The I_N Integrals and the Ultraviolet Cutoff

The I_N integrals defined in (2.20) are certainly the most important functions involved with GEP calculations. By simple power counting one can tell that in (3+1)-dimensions that the I_N integrals diverge. In order to handle this fact, we introduce a regularization device: an ultraviolet cutoff, Λ_{UV}. The idea of the ultraviolet cutoff is to restrict the upper bound of the integration $|\mathbf{k}| \leq \Lambda_{UV}$. By doing so we are able to compare the I_N integrals and, indeed, reparametrize the theory in a manifestly finite form (see Section 2.4). It may seem odd to regularize a theory with such an unphysical construct (why should there be any limit to the momentum integration?). However, by taking $\Lambda_{UV} \to \infty$ we can recover the original theory. Moreover, before making any interpretations of the theory it is imperative that we remove the ultraviolet cutoff.

2.4 The Precarious Reparametrization

The divergence of the GEP arises from the fact that it is expressed in terms of the bare coupling parameters m_B and λ_B, which are themselves not finite. By reparametrizing the theory we can express $\bar{V}_G(\phi_0)$ in a manifestly finite form. It is important to note that the reparametrization has absolutely no effect on the physics [5] and is not nearly the subtle issue of true renormalization (in which the field is rescaled). A convenient and physically meaningful set of "real" parameters are defined below:

$$m_R^2 \equiv \left.\frac{d^2\bar{V}_G}{d\phi_0^2}\right|_{\phi_0=0} \qquad \text{and} \qquad \lambda_R \equiv \left.\frac{1}{4!}\frac{d^4\bar{V}_G}{d\phi_0^4}\right|_{\phi_0=0} \tag{2.26}$$

[3]Note there may be more than one solution to (2.24). One must be sure to select a minimum and not a maximum of V_G.

[4]Note that the I_N integrals have the implicit argument $\bar{\Omega}$ which are in themselves dependent on ϕ_0.

[5]This is because the GEP is exactly renormalization-group (RG) invariant. In other "effective" potentials (*i.e.* the 1LEP) this is not true and there exists a "renormalization-scheme-dependence problem" [1]

Indeed, the "real" parameter m_R can be identified with the mass of a one-particle excitation in the $\phi_0 = 0$ vacuum [7, 8]. In order to evaluate the derivatives of $\bar{V}_G(\phi_0)$ we will need to examine $\bar{\Omega}$'s dependence on ϕ_0. This can be easily achieved with the aid of Equation 2.24.

$$\frac{d\bar{\Omega}}{d\phi_0} = \frac{\phi_0}{\bar{\Omega}} \frac{12\lambda_B}{(1 + 6\lambda_B I_{-1})} \tag{2.27}$$

Evaluating the real parameters with the aid of the above relation and (2.23) we obtain

$$m_R^2 = m_B^2 + 12\lambda_B I_0(\bar{\Omega}_0) = \bar{\Omega}_0^2, \tag{2.28}$$

where $\bar{\Omega}_0$ is the solution of $\bar{\Omega}$ equation at $\phi_0 = 0$, and

$$\lambda_R = \lambda_B \frac{1 - 12\lambda_B I_{-1}(m_R)}{1 + 6\lambda_B I_{-1}(m_R}. \tag{2.29}$$

Both of these equations can be rearranged in order to substitute the real parameters in place of the bare parameters. For m_B this process is obvious; however, for λ_B we obtain

$$\lambda_B = \lambda_R \frac{(1 - 6\lambda_R I_{-1})}{24\lambda_R I_{-1}} \left[1 \pm \left[1 - \frac{48\lambda_R I_{-1}}{(1 - 6\lambda_R I_{-1})^2} \right]^{1/2} \right]. \tag{2.30}$$

One of the solutions yields $\lambda_B = -\lambda_R/2 + O(1/I_{-1})$ which does not give any physically significant situation [6]. The other solution, however, is negative, infinitesimal, and of much interest.

$$\lambda_B = -\frac{1}{6I_{-1}(m_R)} \left[1 + \frac{1}{2\lambda_R I_{-1}(m_R)} + \cdots \right] \tag{2.31}$$

Now with a manifestly finite "effective" potential, we can study the dynamics of the field. Interestingly, the $\bar{\Omega}$ equation does not always have solutions. It does, however, gives us a local minimum and maximum of V_G. Figure 5 illustraits the shape of $\bar{V}_G$. increases, these two solutions coalesce and there cease to be any solutions for the $\bar{\Omega}$ equation. In fact, shortly before the $\bar{\Omega}$ equation looses solutions, $\phi_0^2 = \phi_{0,\text{crit}}^2$, the "end point" of the GEP becomes the global minimum, $V_G(\phi_0, \Omega = 0) < \bar{V}_G(\phi_0)$ for $\phi_0^2 > \phi_{0,\text{break}}^2$. Furthermore, as the coupling becomes stronger, $\phi_{0,\text{break}}^2 \to 0$. Stevenson found that for couplings stronger than $-\lambda_R = 8\phi^2$ the GEP is given entirely by a constant that results from the $\Omega = 0$ end point. The interpretation of these results is that the field can exist in two phases. The first phase, corresponding to weak coupling at low temperature, is an ordinary field theory with massive particles interacting via an attractive force. The second phase, is a strong coupling or high temperature situation which in some enigmatic way corresponds to a massless, essentially free field theory.

[6] Indeed, Stevenson showed any finite λ_B can be disregarded [2].

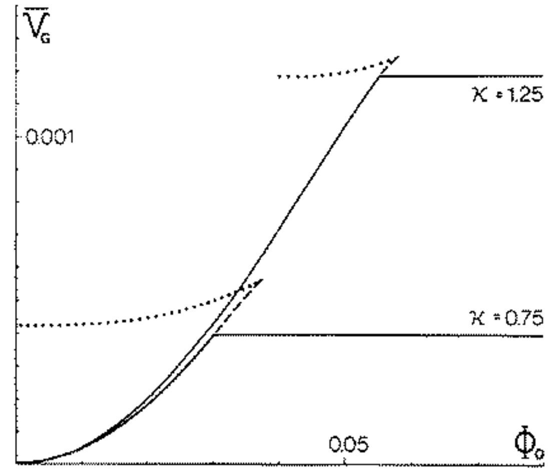

Figure 5: Stevenson's image of the GEP for "precarious ϕ^4 in 3+1 dimensions, with $\kappa = -4\pi^2/\lambda_R = 0.75, 1.25$. The dashed and dotted curves correspond to solution sof the $\bar{\Omega}$ equation which are local-but-not-global minima, and maxima of $V_G(\phi_0, \Omega)$, respectively.

2.5 The Autonomous Renormalization

There exists an alternate renormalization scheme, proposed by Stevenson and Terrach which utilizes an infinite wave function renormalization [3]. In this scheme the bare parameters are chosen such that

$$\phi_B^2 = I_{-1}(\mu)\phi_R^2 \tag{2.32}$$

$$\lambda_B = \frac{1}{12 I_{-1}(\mu)} \tag{2.33}$$

$$m_B^2 = -12\lambda_B I_0(0) + \frac{3m_R^2}{2I_{-1}(\mu)} \tag{2.34}$$

where ϕ_R and m_R are the renormalized field and mass respectively. This scheme predicts a non-zero vacuum expectation value of ϕ_R, which provides SSB needed for the Higgs mechanism. In this sense the autonomous theory is directly applicable to the Standard Model Higgs. Indeed the predicted mass of the Higgs boson is given by

$$m_h^2 = \left.\frac{d^2\bar{V}_G}{d\phi_R^2}\right|_{\phi_R = V} = 8\pi^2 V^2 + 2m_R^2, \tag{2.35}$$

where V is the vacuum expectation value of the Higgs field (*i.e.* the global minimum of $\bar{V}_G$. The parameter m_R^2 can be either positive or negative, leaving the Higgs mass as a free parameter. The parameter V is obtained from the Fermi Constant ($V = (\sqrt{2}G_F)^{-1/2} \approx$ 246 GeV) sets the natural scale for the Higgs mass at $m_h \sim 2.2$ TeV. As m_R^2 becomes large and negative, the GEP more closely mirrors the results of perturbation theory. Though there is no lower limit on the Higgs mass, once $m_R^2 > 16\pi^2 V^2$ the vacuum expectation value of the Higgs field becomes zero. When there ceases to be any SSB there is no longer a Higgs mechanism.

Thus, this theory predicts $m_h < 4.4$ TeV. Appendix A is dedicated to the most significant scenarios involving different Higgs mass and experimental results.

3 The BCS-Generalization

Yotsuyanagi proposed a generalization of the GEP motivated by superconductor theory formulated by John Bardeen, Leon Cooper, and Robert Schrieffer, the BCS theory. Just as in the case of the Gaussian effective potential, the BCS-generalization is an approximation of V_{eff} (1.8) obtained by restricting $|\psi\rangle$ to a subspace of possible field configurations. The generalization was presented by I. Yotsuyanagi in [4]. The generalization is motivated from the BCS-theory of solid-state physics. Essentially, the configuration of the trial field is now not only parametrized by Ω, but also by the function $v_k = v(|\mathbf{k}|)$. The Gaussian effective potential is a special case of the BCS-generalization corresponding to $v_k \equiv 0$.

Again we start with the Lagrangian density [7]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4}\phi^4, \tag{3.1}$$

we expand the field about ϕ_0

$$\phi = \phi_0 + \int (dk)_\Omega[a_\Omega(\vec{k})e^{-ik.x} + a_\Omega^\dagger(\vec{k})e^{ik.x}], \tag{3.2}$$

and evaluate

$$\partial_\mu\phi = \int (dk)_\Omega(-ik_\mu)[a_\Omega(\mathbf{k})e^{-ik.x} - a_\Omega^\dagger(\mathbf{k})e^{ik.x}]. \tag{3.3}$$

This yields the corresponding Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4}\phi^4, \tag{3.4}$$

where $k.x \equiv k^\mu x_\mu = \omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x}$ and

$$k_0 = \omega_{\mathbf{k}}(\Omega) \equiv (\mathbf{k}^2 + \Omega^2)^{1/2}. \tag{3.5}$$

3.1 The Ansatz

We assume the BCS-motivated trial state

$$|\tilde{0}\rangle_{\phi_0,\Omega,v_k} = \prod_{k>0}\left(1 + \frac{v_k}{u_k}a_k^\dagger a_{-k}^\dagger\right)|0\rangle_{\phi_0,\Omega}, \tag{3.6}$$

where

$$u_k^2 + v_k^2 = 1. \tag{3.7}$$

Again, $|0\rangle_{\phi_0,\Omega}$ is the vacuum used in the GEP. The ϕ_0, Ω subscripts will be left off to avoid clutter; however, it is important to realize $|0\rangle$'s dependence on both parameters. The state $|\tilde{0}\rangle$ can be thought of (especially with the help of (3.10)) as a superposition of all combinations

[7] Note a change in the normalization of the bear coupling constant $\lambda_0 = 4\lambda_B$. This is done to be consistent with the major sources.

of mode-"antimode" pairs. Because $|\tilde{0}\rangle$ is not normalized we are forced to reexpress (2.10) as

$$W(\phi_0, \Omega; v_k) \equiv \frac{\langle\tilde{0}|H|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle}. \tag{3.8}$$

In order to obtain an "effective" potential from W we need to minimize $W(\phi_0, \Omega; v_k)$ with respect to the variational parameters Ω and v_k.

3.2 The Variational Parameters

Now that our energy expectation value of a given trial field is a function (functional) of two variational "parameters" we must rethink the minimization procedure. This is not difficult, having its roots in basic multivariate calculus. Ignoring, for the moment, the issues brought up in Section 1.5, all we need to do in order to find a local minimum of $W(\phi_0, \Omega; v_k)$ is require the partials of W with respect to the variational "parameters" are simultaneously zero. In general, this will give us a set of equations which we can use to optimize our variational parameters. For Ω this is straightforward; however, for v_k this requires a functional minimization. Once minimized, we can move right along reparametrizing the theory and studying its dynamics.

Returning to the subtlety in the *global* minimization of W, before we are done we must look at the "end-point" contributions of our variational parameters. For Ω this corresponds to $\Omega = 0$ and $\Omega \to \infty$; however, for v_k this is not so clear. With no restrictions on v_k there are no "end-points" to consider. If, however, restrictions are placed on v_k (*e.g.* $v_k \in \mathcal{C}^1$), then the notion of an "end-point" may be relevant. Furthermore, because this "end-point" corresponds to the limit of a sequence of functions it is not an easy task to study. Fortunately, because of the nature of variational methods, an approaching function will most likely illustrate whatever physics is to be found in the limit.

One of our parameters, Ω, is already familiar from the GEP. Our other "parameter" v_k is a function which serves as a sort of recipe for how to construct our trial state from the Hartree-Fock vacuum. From (3.6) it is clear that $v_k \equiv 0$ reduces the BCS-generalization to the Gaussian case. In other words, the Gaussian is just a special case of the BCS-generalization.

3.3 Evaluating $W(\phi_0, \Omega; v_k)$

In (3.6) the field is utilizing a "box normalization"[8] in which the momenta have discrete values. Thus (3.6) can be written explicitly as

$$|\tilde{0}\rangle = \left(1 + \frac{v_{k_1}}{u_{k_1}} a^\dagger_{k_1} a^\dagger_{-k_1}\right)\left(1 + \frac{v_{k_2}}{u_{k_2}} a^\dagger_{k_2} a^\dagger_{-k_2}\right)\left(1 + \frac{v_{k_3}}{u_{k_3}} a^\dagger_{k_3} a^\dagger_{-k_3}\right)\cdots|0\rangle \tag{3.9}$$

[8] The box normalization should be familiar from quantum mechanics. By applying boundry conditions to the box we obtain a discrete set of momentum modes. Before any physics is done the limit of the infinite sized box is taken producing a continuous momentum spectrum.

After distributing terms it can be shown that

$$\prod_{k>0}(1+\alpha_k) = 1 + \sum_i \alpha_i + \sum_{i,j;i\neq j} \alpha_i\alpha_j + \sum_{i,j,k;i\neq j\neq k} \alpha_i\alpha_j\alpha_k + \cdots. \tag{3.10}$$

In this form (3.8) can be evaluated term by term.

First we look at the creation and annihilation operators on $|\tilde{0}\rangle$. Where before $a|0\rangle = 0$, now the vacuum is rich with mode-"antimode" pairs. Let us first introduce some notation conventions: $\alpha_i = v_i/u_i$, $|p\rangle = a_p^\dagger a_{-p}^\dagger|0\rangle$, $|i,j\rangle = |j,i\rangle = a_j^\dagger a_{-j}^\dagger|i\rangle$, and $|p_2\rangle = a_p^\dagger a_{-p}^\dagger|p\rangle$.

$$\begin{aligned}
\langle\tilde{0}|a_p^\dagger a_{p'}^\dagger|\tilde{0}\rangle &= \delta_{p',-p}\langle\tilde{0}|a_p^\dagger a_{-p}^\dagger|\tilde{0}\rangle \\
&= \delta_{p',-p}\langle\tilde{0}|\left[|p\rangle + \sum_{i;i\neq p}\alpha_i|i,p\rangle + \alpha_p|p_2\rangle + \sum_{i,j;i\neq j\neq p}\alpha_i\alpha_j|i,j,p\rangle + \sum_{i;i\neq p}\alpha_i\alpha_p|i,p_2\rangle + \cdots\right] \\
&= \delta_{p',-p}\left[\alpha_p + \sum_{i;i\neq p}(\alpha_p\alpha_i)\alpha_i + \sum_{i,j;i\neq j\neq p}(\alpha_i\alpha_j\alpha_p)\alpha_i\alpha_j + \cdots\right] \\
&= \delta_{p',-p}\frac{v_p}{u_p}\prod_{k>0;k\neq p}\left[1+\frac{v_k^2}{u_k^2}\right]
\end{aligned} \tag{3.11}$$

Note that if $p' \neq -p$ [9] , then the $a_p^\dagger a_{p'}^\dagger|\tilde{0}\rangle$ will nowhere be a superposition of mode-"antimode" pairs leaving no amplitude in the projection onto $|\tilde{0}\rangle$. In a similar way, $\langle\tilde{0}|a_p a_{p'}|\tilde{0}\rangle$ and $\langle\tilde{0}|a_p a_{p'}^\dagger|\tilde{0}\rangle$ can be evaluated. However, before W can be evaluated we must find it's normalization.

$$\langle\tilde{0}|\tilde{0}\rangle = 1 + \sum_i \alpha_i^2 + \sum_{i,j;i\neq j}\alpha_i\alpha_j + \cdots = \prod_{k>0}\left[1+\frac{v_k^2}{u_k^2}\right] = \prod_{k>0}\frac{1}{u_k^2} \tag{3.12}$$

Now using the normalization constant, $\langle\tilde{0}|\tilde{0}\rangle^{-1}$ we arrive at

$$\frac{\langle\tilde{0}|a_p^\dagger a_{p'}^\dagger|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \frac{\langle\tilde{0}|a_p a_{p'}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = v_p u_p \delta_{p',-p} \tag{3.13}$$

In the case of $\langle\tilde{0}|a_p a_{p'}^\dagger|\tilde{0}\rangle$, the p' selection criteria is different. Because only one excitation of a mode is possible, the only way to have a non-zero amplitude upon projection is for the annihilation operator to remove anything that was created (*i.e.* $p = p'$). Thus we obtain

$$\frac{\langle\tilde{0}|a_p a_{p'}^\dagger|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = (v_p^2+1)\delta_{p,p'} \tag{3.14}$$

and with help from the commutation relation $[a_p, a_{p'}^\dagger] = \delta_{p,p'}$ we obtain

$$\frac{\langle\tilde{0}|a_p^\dagger a_{p'}|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = v_p^2\delta_{p,p'}. \tag{3.15}$$

[9]In the limit of the box normalization, when the allowed momenta are no longer discrete, we will have $\delta_{p,p'} \to \delta(p-p')$ (*i.e.* a delta function not a Kronecker delta).

Moving right ahead we can now evaluate W term by term. For the kinetic energy term we have [10]

$$\frac{\langle\tilde{0}|\dot{\phi}^2|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \int (dk)\omega_k^2[2v_k^2 + 1 - 2v_k u_k] \tag{3.16}$$

and for the spatial part [11]

$$\begin{aligned}\frac{\langle\tilde{0}|(\nabla\phi)^2|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} &= \int (dk)|\mathbf{k}|^2[2v_k^2 + 1 + 2v_k u_k] \\ &= \int (dk)\omega_k^2[2v_k^2 + 1 + 2v_k u_k] - \int (dk)\Omega_k^2[2v_k^2 + 1 + 2v_k u_k]\end{aligned} \tag{3.17}$$

For the potential we have

$$\frac{\langle\tilde{0}|\phi^2|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \phi_0^2 + 2\int (dk)[v_k u_k + 2v_k^2 + 1] \tag{3.18}$$

and

$$\frac{\langle\tilde{0}|\phi^4|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \phi_0^4 + 6\phi_0^2\int (dk)[1 + u_k^2 + v_k^2] + 3\int (dk)[1 + u_k^2 + v_k^2]\int (dk')[1 + u_{k'}^2 + v_{k'}^2] \tag{3.19}$$

With the notation

$$\Delta_0(\Omega) \equiv \int 2(dk)_\Omega(u_k v_k + v_k^2) \tag{3.20}$$

$$\Delta_1(\Omega) \equiv \int 2(dk)_\Omega\omega_k^2 v_k^2 \tag{3.21}$$

and a bit of algebra we obtain

$$\begin{aligned}W(\phi_0,\Omega;v_k) &= I_1(\Omega) + \frac{(m_0^2 - \Omega^2)}{2}I_0(\Omega) + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4}\phi_0^4 + \frac{3\lambda_0}{4}[I_0^2(\Omega) + 2\phi_0^2 I_0(\Omega)] \\ &+ \Delta_1(\Omega) + \Delta_0(\Omega)\left[\frac{(m_0^2 - \Omega^2)}{2} + \frac{3\lambda_0}{2}(\phi_0^2 + I_0(\Omega)) + \frac{3\lambda_0}{4}\Delta_0(\Omega)\right]\end{aligned} \tag{3.22}$$

The I_N integrals are not with out physical meaning. Their origin comes from integrating the structure of the field. It is not surprising that some quantities are infinite in a system consisting of a continuously infinite number of coupled harmonic oscillators. In fact, the energy expectation value of a free field is infinite resulting from the zero-point energy of each quantum oscillator. [12] We would expect that with the transition $|0\rangle \to |\tilde{0}\rangle$ the physical concept associated with the I_N integrals must also generalize. Indeed this can be formalized with the observation

$$I_0 \to I_0 + \Delta_0 \quad \text{and} \quad I_1 \to I_1 + \Delta_1. \tag{3.23}$$

[10]In this evaluation we have two integrals and we use the fact that $\omega_k = \omega_{-k}$ to obtain the $+\omega_k^2$ term.

[11]In this evaluation we have two integrals and we use the fact that $-\mathbf{k}\cdot\mathbf{k}' \to -|\mathbf{k}|^2$ and not $|\mathbf{k}|^2$ as one might write in haste.

[12]This is, of course, not a problem because only energy differences can be measured.

In fact, with a little rearrangement we can write (3.22) as its Gaussian equivalent (2.21) with the above substitution.

$$\begin{aligned} W(\phi_0, \Omega; v_k) &= [I_1(\Omega) + \Delta_1(\Omega)] + \frac{(m_0^2 - \Omega^2)}{2}[I_0(\Omega) + \Delta_0(\Omega)] + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4}\phi_0^4 \quad (3.24) \\ &+ \frac{3\lambda_0}{2}\phi_0^2[I_0(\Omega) + \Delta_0(\Omega)] + \frac{3\lambda_0}{4}[I_0(\Omega) + \Delta_0(\Omega)]^2 \end{aligned}$$

Because of this observation, it is one of the immediate goals to study Δ_0 and Δ_1 to see if they behave in a manner similar to the I_N integrals.

3.4 Optimization of W

This is undoubtedly the most significant functional step in any variational approach. It is absolutely crucial that this step by done with the greatest of care. First, we must minimize W with respect to each parameter holding the other parameter constant. Next, we apply the two independent optimization conditions simultaneously and study the theory. Before making any conclusions, we must consider the end points of each parameter. With no restrictions on v_k this means studying the theory with v_k optimized and $\Omega = 0$ and $\Omega \to \infty$.

3.4.1 Minization With Respect To Ω

This minimization is straighforward just as it was in the Gaussian case. In the Gaussian case, however, it was possible to simplify the derivative of the I_N integrals. Furthermore, it is important to keep in mind that v_k is held constant throughout this minimiziation. Because the Δ_N integrals depend on, as of yet, undetermined functions, it is not possible to simplify those integrals. After differentiating (3.24) and sorting terms we arrive at

$$\frac{\partial W}{\partial \Omega} = -F\frac{\partial}{\partial \Omega}(\Delta_0 + I_0) + \frac{\partial \Delta_1}{\partial \Omega} - \Omega\Delta_0, \tag{3.25}$$

where

$$F = \frac{(\Omega^2 - m_0^2)}{2} - \frac{3\lambda_0}{2}(\phi_0^2 + I_0 + \Delta_0). \tag{3.26}$$

Interestingly, F is very reminiscent of the "$\bar{\Omega}$-equation" for the GEP (2.24). In the case where Ω is given by $\bar{\Omega}$ and the substitution $I_0 \to I_0 + \Delta_0$ is valid, $F = 0$. This observation will be investigated in Section 3.5.

3.4.2 Stationarization With Respect To v_k

The minimization with respect to Ω was only half of the battle. Now we need to find the function v_k (and thus u_k) which minimize W. This is a calculus of variations problem. Because functional differentiation may be unfamiliar to the reader, the derivation is given explicitly. Again we note that Ω is treated as a constant in this minimization.

$$
\begin{aligned}
\frac{\delta\Delta_0}{\delta v_k} &= \frac{\delta}{\delta v_k}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}(u_p v_p + v_p^2) \\
&= \frac{\delta}{\delta v_k}\frac{1}{2\pi^2}\int dpp^2\frac{1}{\omega_p}(u_p v_p + v_p^2) \\
&= \frac{1}{2\pi^2}\int dpp^2\frac{1}{\omega_p}\left[u_p + \frac{v_p^2}{u_p} + 2v_p\right]\delta(p-k) \\
&= \frac{k^2}{2\pi^2}\frac{1}{\omega_k}\left[\frac{1-2v_k^2}{u_k} + 2v_k\right]
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
\frac{\delta\Delta_1}{\delta v_k} &= \frac{\delta}{\delta v_k}\int \frac{d^3p}{(2\pi)^3}\omega_p v_p^2 \\
&= \frac{\delta}{\delta v_k}\frac{4\pi}{(2\pi)^3}\int dpp^2\omega_p v_p^2 \\
&= \frac{1}{2\pi^2}\int dpp^2 \cdot 2v_p\omega_p\delta(p-k) \\
&= \frac{k^2}{2\pi^2}2\omega_k v_k
\end{aligned} \tag{3.28}
$$

Substituting this into $\delta W/\delta v_k$ we obtain

$$
\frac{\delta W}{\delta v_k} = \frac{k^2}{2\pi^2\omega_k}\underbrace{\left[2v_k\omega_k^2 - \left(\frac{1-2v_k^2}{u_k} + 2v_k\right)F\right]} \tag{3.29}
$$

which again shows dependence on the combination F.

3.5 Reduction of Generality

The Ω optimization condition given in Equation 3.25 requires

$$
F\frac{\partial}{\partial\Omega}(\Delta_0 + I_0) = \frac{\partial\Delta_1}{\partial\Omega} - \Omega\Delta_0. \tag{3.30}
$$

Evaluating the left hand side we obtain

$$
\begin{aligned}
F\frac{\partial}{\partial\Omega}(\Delta_0 + I_0) &= F\frac{\partial}{\partial\Omega}\int(dk)[2u_k v_k + 2v_k^2 + 1] \\
&= -F\Omega\int(dk)\frac{1}{\omega_k^2}[2u_k v_k + 2v_k^2 + 1].
\end{aligned} \tag{3.31}
$$

Similarly, for the right side we obtain

$$
\begin{aligned}
\frac{\partial\Delta_1}{\partial\Omega} - \Omega\Delta_0 &= \frac{\partial\Delta_1}{\partial\Omega}\int(dk)\omega_k^2 2v_k^2 - 2\Omega\int(dk)[u_k v_k + v_k^2] \\
&= 2\Omega\int(dk)v_k^2 - 2\Omega\int(dk)[u_k v_k + v_k^2] \\
&= -2\Omega\int(dk)u_k v_k
\end{aligned} \tag{3.32}
$$

Putting these two together and using the v_k optimization condition given in Equation 3.29 we find

$$\begin{aligned}
\Omega \int (dk) \frac{1}{\omega_k^2}[2u_kv_k + 2v_k^2 + 1]F &= \Omega \int (dk) 2u_kv_k \\
\Omega \int (dk) \frac{1}{\omega_k^2}[2u_kv_k + 2v_k^2 + 1]F &= \Omega \int (dk) \frac{1}{\omega_k^2} 2u_kv_k\omega_k^2 \\
\Omega \int (dk) \frac{1}{\omega_k^2}[2u_kv_k + 2v_k^2 + 1]F &= \Omega \int (dk) \frac{1}{\omega_k^2}(1 - 2v_k^2 + 2u_kv_k)]F \\
4F\Omega \int (dk) \frac{v_k^2}{\omega_k^2} &= 0
\end{aligned} \tag{3.33}$$

Excluding the case $\Omega = 0$, we either have $F = 0$ [13] or

$$\int (dk) \frac{v_k^2}{\omega_k^2} = 0 \qquad \Longrightarrow \qquad v_k \equiv 0 \qquad (\forall k) \tag{3.34}$$

That is to say, in order to satisfy both optimization conditions simultaneously the optimal field configuration corresponds to the Gaussian case. This phenomena can be thought of in two different ways. The first is that the optimization conditions reduce the generality of our ansatz bringing us full circle to the GEP. [14] The second way to think of this result as an expansion in generality of the GEP. In other words, the GEP is the optimal approximation to "the effective potential" in a larger subspace of possible field configurations than was previously realized. In either case, we have the important result that $W(\phi_0, \bar{\Omega}; \bar{v}_k) = V_G(\phi_0, \bar{\Omega})$. An observant reader may have noticed that we have treated v_k and u_k in $|\tilde{0}\rangle$ as real valued functions.

3.6 The End Point Contributions

By moving to the end-point of Ω we loose the the optimization condition of Ω given by (3.25). We are, however, left with the stationarity condition (3.29). Thus we have

$$2v_k(\omega_k^2 - F) = \frac{1 - 2v_k^2}{u_k} F, \tag{3.35}$$

By squaring both sides[15] and using the v_k-u_k relationship in (**??**) we obtain

$$v_k^4 - v_k^2 + \frac{F^2}{4\eta} = 0, \tag{3.36}$$

where

$$\eta = (\omega_k^2 - F)^2 + F^2. \tag{3.37}$$

[13] $F = 0$ corresponds to $v_k \equiv 0$, the Gaussian case, as we will show in Section 3.6

[14] Interestingly, if one assumes that v_k is complex, then the minimization condition fixes $\Im(v_k) = 0$ - reducing the generality of our ansatz.

[15] The reader should keep in mind the loss of sign information in our solutions for v_k resultant from this step

The solutions are

$$v_k^2 = \frac{1 \pm \sqrt{1-D^2}}{2} \qquad \text{and} \qquad u_k^2 = \frac{1 \mp \sqrt{1-D^2}}{2}, \tag{3.38}$$

where

$$D^2 = \frac{F^2}{\eta} = \frac{F^2}{(\omega_k^2 - F)^2 + F^2}. \tag{3.39}$$

In order for $|\tilde{0}\rangle$ to be defined for all F we must have $u_k \neq 0$ for all possible F. This fixes the signs, thus

$$v_k^2 = \frac{1 - \sqrt{1-D^2}}{2} \qquad \text{and} \qquad u_k^2 = \frac{1 + \sqrt{1-D^2}}{2}. \tag{3.40}$$

As advertised, $F = 0 \rightarrow D = 0$ which corresponds with $v_k = 0$ (a.k.a the GEP).

There still remains a sign issue when we look at explicit solutions for v_k (u_k). To find this sign we require that v_k (u_k) be a solution to the original condition (3.35). In order to evaluate (3.35) we need the term

$$v_k u_k = \pm \frac{\sqrt{D^2}}{2} \tag{3.41}$$

with $\pm$ corresponding to the sign of $v_k u_k$. We've gotten ourselves in a bit of a loop - now have to worry about the sign of D as well. This is fine, our logic will be that we *define D to be the positive root of* (3.39) keeping the sign ambiguities solely in $v_k u_k$. In order for $v_k u_k$ to remain a solution to (3.35) we need

$$v_k u_k = \frac{\varepsilon(\omega_k^2 - F)F}{2\sqrt{\eta}} \tag{3.42}$$

where $\varepsilon(x) = \{+1 \text{ for } x > 0 \text{ and } -1 \text{ for } x < 0\}$. So the sign of $v_k u_k$ is positive for $0 < F < \omega_k^2$ and negative otherwise. For large k, $v_k u_k$ is positive.

It is useful to be aware of the relationships of v_k and u_k. From (3.7) it is clear that v_k and u_k can be thought of as components to the unit circle parametrized by θ_k. Let

$$v_k = \sin\theta_k \qquad \text{and} \qquad u_k = \cos\theta_k \tag{3.43}$$

so that $\theta_k = 0$ corresponds to the GEP. With the help of (3.40) we find the following relationships

$$\cos 2\theta_k = \frac{\omega_k^2 - F}{\sqrt{\eta}} = \frac{\omega_k^2 - F}{\sqrt{(\omega_k^2 - F)^2 + F^2}} \tag{3.44}$$

$$\sin 2\theta_k = \frac{F}{\sqrt{\eta}} = \frac{F}{\sqrt{(\omega_k^2 - F)^2 + F^2}} \tag{3.45}$$

$$\tan 2\theta_k = \frac{F}{\omega_k^2 - F} \tag{3.46}$$

Now that we have the form of v_k and u_k we can, in principle, evaluate Δ_0 and Δ_1.

$$\begin{aligned}\Delta_0(\Omega) &\equiv \int 2(dk)_\Omega(u_k v_k + v_k^2) \\ &= \frac{1}{2\pi^2}\int_0^{\Lambda_{UV}} dk k^2 \frac{1}{\omega_k}\left[\frac{\varepsilon(\omega_k^2 - F)F}{2\sqrt{(\omega_k^2-F)^2+F^2}} + 1 - \frac{|\omega_k^2 - F|}{\sqrt{(\omega_k^2-F)^2+F^2}}\right]\end{aligned} \tag{3.47}$$

$$\begin{aligned}\Delta_1(\Omega) &\equiv \int 2(dk)_\Omega \omega_k^2 v_k^2 \\ &= \frac{1}{2\pi^2}\int_0^{\Lambda_{UV}} dk k^2 \omega_k\left[1 - \frac{|\omega_k^2 - F|}{\sqrt{(\omega_k^2-F)^2+F^2}}\right]\end{aligned} \tag{3.48}$$

By simple power counting we can see that both Δ_0 and Δ_1 are logarithmically divergent. The Δ_N integrals are less divergent than their I_N counterparts. Unlike the I_N integrals in which the divergences get worse as N increases, the divergences in Δ_0 and Δ_1 are of the same order. There is a neat trick which allows us to reduce the divergences in many expressions. By taking the difference of a divergent integral with another divergent integral of the same order we are left with a difference of lower order. Thus, the difference of two logarithmically divergent integrals is a number $(\ln(\Lambda/\mu) - \ln(\Lambda/\nu) = \ln(\nu/\mu))$. This motivates the following two combinations

$$\Delta_0(\Omega) - F I_{-1}(F) \qquad \text{and} \qquad \Delta_1(\Omega) - \tfrac{1}{2}F^2 I_{-1}, \tag{3.49}$$

which will be of use in Section 3.6.1.

3.6.1 The Precarious Reparametrization

The most important result from the optimization of W in Section 3.5 is that the reparametrization of the BCS-generalization proceeds in full accord with the GEP. Because the real parameters are evaluated at the origin, the end point solutions do not contribute. Thus, we have

$$\lambda_0 \cong -\frac{2}{3I_{-1}(m_R)} - \frac{4}{3I_{-1}^2(m_R)\lambda_R} \tag{3.50}$$

and

$$m_R^2 = m_0^2 + 3\lambda_0 I_0(m_R). \tag{3.51}$$

By substituting the real parameters for the bare parameters into F we obtain

$$\Delta_0(\Omega) - F I_{-1}(m_R) = -\phi_0^2 + \frac{\Omega^2 - m_R^2}{\lambda_R} - \frac{2\Delta_0(\Omega)}{\lambda_R I_{-1}(m_R)} - \frac{m_R^2 L_2(\Omega^2/m_R^2)}{16\pi^2} \tag{3.52}$$

Using the relation $I_{-1}(F) - I_{-1}(m_R) = -(1/8\pi^2)\ln(F/m_R^2)$ we can rewrite (3.52) to obtain a relationship between a particularly simple combination on the left and the field modulo a constant (on the right). This form is also superior because it is F that is setting the scale for the physics. This will be very useful in our analysis of the behavior of $W(\phi_0, \Omega = 0; v_k)$.

$$\Delta_0(\Omega) - F I_{-1}(F) = -\phi_0^2 + \frac{\Omega^2 - m_R^2}{\lambda_R} - \frac{2\Delta_0(\Omega)}{\lambda_R I_{-1}(m_R)} - \frac{m_R^2 L_2(\Omega^2/m_R^2)}{16\pi^2} + \frac{F}{8\pi^2}\ln(F/m_R^2) \tag{3.53}$$

As ϕ_0 becomes arbitrarily large (3.53) reduces to

$$\phi_0^2 = \frac{F}{8\pi^2}\ln(F/m_R^2) \tag{3.54}$$

We now have the large-ϕ_0 behavior of F. In order to evaluate $W(\phi_0, \Omega = 0; v_k)$, we need to develop some tools. Recall the motivation for(3.49), by taking the difference of logarithmically divergent integrals, we are left with a constant. At the $\Omega = 0$ end point we can evaluate (3.49) with the aid of the substitution $k^2 \to |F|x$. This leads to

$$\begin{aligned}
\Delta_0(0) - FI_{-1}(F) &\equiv \int (dk)_\Omega (2u_k v_k + 2v_k^2 - \frac{F}{\omega_k^2}) \qquad (3.55)\\
&= \frac{|F|}{8\pi^2}\int_0^{\frac{\Lambda_{UV}^2}{|F|}} dx \left[1 - \frac{\varepsilon(x-\varepsilon(F))(x-2\varepsilon(F))}{\sqrt{(x-\varepsilon(F))^2+1}} - \frac{\varepsilon(F)}{x\left(1+\frac{\varepsilon(F)}{x}\right)^{3/2}}\right]\\
&= \frac{|F|}{8\pi^2}\Theta_0\\
\Delta_1(0) - \frac{1}{2}F^2 I_{-1}(F) &\equiv \int (dk)_\Omega \left(2\omega_k^2 v_k^2 - \frac{F^2}{2\omega_k^2}\right) \qquad (3.56)\\
&= \frac{|F|}{8\pi^2}\int_0^{\frac{\Lambda_{UV}^2}{|F|}} dx \left[x\left(1 - \frac{|x-\varepsilon(F)|}{\sqrt{(x-\varepsilon(F))^2+1}}\right) - \frac{1}{2x\left(1+\frac{\varepsilon(F)}{x}\right)^{3/2}}\right]\\
&= \frac{F^2}{8\pi^2}\Theta_1
\end{aligned}$$

where Θ_0 and Θ_1 are finite constants. Using (3.54), (3.55), and (3.56) in $W(\phi_0, \Omega; v_k)$ at the Ω end point we obtain the large-ϕ_0 solution to W.

$$W(\phi_0, \Omega = 0) = -\frac{4\pi^2\phi_0^4}{\log(\phi_0^2/m_R^2)}\left(1 + O\left(\frac{\log\log\phi_0^2}{\log\phi_0^2}\right)\right), \tag{3.57}$$

which is unbounded from below. This suggests, in accordance with Yotsuyanagi's conclusions, that ϕ^4 QFT with $\lambda_0 \to 0_-$ is not infinitely metastable, but instead, unstable even when the ultraviolet cutoff is removed.

3.6.2 The Autonomous Renormalization

In the autonomous renormalization Yotsuyanagi found that the end point contributions did not provide a global minimum to W, thus both minimization conditions must be satisfied simultaneously. As we have already seen, if both optimization conditions are met simultaneously we obtain the GEP. From this realization it is not surprising that Yotsuyanagi's results agreed with that of the Gaussian case. Furthermore, this finding supports the claim that the GEP is "effectively exact" in the autonomous case [3].

4 Summary, Outlook, and Conclusions

4.1 Summary

The BCS-generalization of the GEP provides additional insight into both the generality of the GEP and the behavior of $\lambda\phi^4$ scalar quantum field theory. In particular, it was found that the GEP is the optimal solution to a broader subspace of possible field configurations, corresponding to the BCS ansatz, than was previously known. The added generality supports the use of the GEP as a field-theoretic tool. Specifically it supports the claim that it is "effectively exact" in the autonomous case. The realization that the GEP is the optimal BCS-type trial state, in the autonomous case, leads to Yotsuyanagi's conclusion in a direct and elucidating way. In the precarious case, the BCS-generalization reveals an instability in the field that the GEP could not see. We arrive at the same conclusion as Yotsuyanagi with clarification on subtle sign issues.

4.2 Outlook

The instability of $\lambda\phi^4$ QFT in the precarious reparameterization scheme gives way to a plethora of questions. The most immediate goal is to characterize the $W(\phi_0, \Omega; v_k)$. For instance:

- How does W behave for small ϕ_0?
- Does the $(\Omega = 0, v_k \neq 0)$ end point solution take over before or after the $\bar{\Omega}$ equation looses solutions?
- Does the $(\Omega = 0, v_k \neq 0)$ end point solution take over before or after the $\bar{\Omega}$ equation looses solutions?
- Does the $(\Omega = 0, v_k \neq 0)$ end point solution take over before or after the $(\Omega = 0, v_k = 0)$ solutions?

In order to answer these questions the following approach has been devised. Using the stationarity condition for v_k we obtain the form of v_k in terms of the quantity F. By treating F as a free parameter the transition of W from $F = 0$ (a.k.a the GEP, $v_k = 0$,) to $F = \bar{F}$ can be studied.

Variational methods such as the GEP provide a unique, powerful way to study quantum field theories. Despite the attractive features of variational methods, perturbative techniques dominate. In approximately six years, when the LHC turns on at CERN, and possibly earlier at the Tevatron, a great deal will be learned about both methods. If the Higgs is not found at the LHC, then variational methods like the GEP will gain popularity. For a more in depth discussion about the experimental and theoretical scenarios, see Appendix A.

4.3 Conclusions

Where as Stevenson found that the GEP allowed for an infinitely metastable $\lambda\phi^4$ field theory when $\lambda_B \to 0_-$, the Yotsuyanagi's BCS-motivated generalization reveals that the theory is in fact unstable. On the other hand, in the autonomous renormalization scheme ($\lambda_B \to 0_+$) the GEP is itself the optimal BCS-generalization of the GEP. The autonomous theory allows for SSB and is considered to be an "effectively exact" approximation of the true effective potential.

A The Higgs Boson

This Section is meant to provide a setting for this research and elucidate the research's place in that context. Essentially, the context of this research is the Standard Model of particle physics. This is a theory of the fundamental particles and their interactions. One particularly sensitive aspect of the Standard Model is the origin of the masses of the fundamental particles. If the masses for the particles are put in "by hand" the Lagrangian looses its gauge invariance and the theory becomes unrenormalizable. A special particle, the Higgs boson, provides a way to give masses to the particles without spoiling gauge invariance. This process is known as the Higgs mechanism and the basic idea is that if there exists a scalar particle, the Higgs boson, which couples to other fields and the expectation value of the Higgs field is non-zero, then the interaction produces an "effective mass term" in the Lagrangian of the field to which it couples. In order to accomplish this, the symmetry of the Higgs field must be *spontaneously broken*. This is the origin of the $\lambda\phi^4$ term. In the limit of weak coupling with other fields, the Lagrangian of the Higgs is the $\lambda\phi^4$ field theory studied in this thesis.

In addition to the relevance to the Higgs, $\lambda\phi^4$ is interesting on its own. Because the Higgs is a scalar, the theory which describes it is the simplist non-free field theory which can be expressed. Before moving onto more complicated theories, we should be sure we understand the basics. The precarious solution to the GEP is infinitely metastable, however, it does not produce the SSB needed to be directly relevant to the Higgs mechanism. Furthermore, it appears that the precarious state is not even infinitely metastable due to the field configurations found by the BCS method. On the other hand, the autonomous theory is stable and produces SSB. Thus, the autonomous case is directly relevant to the Higgs mechanism.

The autonomous renormalization of the GEP predicted mass of the Higgs boson is given by

$$m_h^2 = \left.\frac{d^2\bar{V}_G}{d\phi_R^2}\right|_{\phi_R=V} = 8\pi^2V^2 + 2m_R^2, \tag{1.1}$$

where V is the vacuum expectation value of the Higgs field (*i.e.* the global minimum of $\bar{V}_G$). The parameter m_R^2 can be either positive or negative, leaving the Higgs mass as a free parameter. The parameter V is obtained from the Fermi Constant ($V = (\sqrt{2}G_F)^{-1/2} \approx$ 246 GeV) sets the natural scale for the Higgs mass at $m_h \sim 2.2$ TeV. Though there is no lower limit on the Higgs mass, once $m_R^2 > 16\pi^2V^2$ the vacuum expectation value of the Higgs field becomes zero. When there ceases to be any SSB, there is no longer a Higgs mechanism. Thus, this theory predicts $m_h < 4.4$ TeV.

The idea of a Higgs mass on the order of 2.2 TeV would bother most high energy particle physicists. The more common quote of the Higgs mass looks something more like

$$m_h = (76^{+85}_{-47} \pm 10)\text{GeV}. \tag{1.2}$$

The strange looking errors come from the minimum of a χ^2 on a logarithmic scale. The second error is due to theoretical uncertainties [12]. These results arise from the standard approach to field theory: perturbation theory. Unlike the GEP, perturbation theory is not Renormalization Group invariant. This implies that there are delicate issues involved with renormalizing peturbative theories. Mass bounds on the Higgs field which appeal to "unitarity" arguments are very common. Unfortunately, these arguments are more of a sign of perturbation theory breaking down than an upperbound on the Higgs mass. Currently the Higgs mass has been excluded with 95% confidence from being less than about 88 GeV. In the near future LEP should be able to achieve center-of-mass energies approaching 200 GeV with Higgs mass sensitivity up to roughly 105 GeV. The Tevatron will be able to achieve a center-of-mass energy of about 2 TeV corresponding to a Higgs mass up to 120 GeV. Unfortunately, for the Tevatron to see the Higgs at 120 GeV it would have to gather 25 fb^{-1} worth of data. In about 2005, the LHC should come on line with a center-of-mass energy of 14 TeV and a luminosity of up to 10^{34} $cm^{-2}s^{-1}$. In principle, the LHC will be able to search for a Higgs as heavy as 1 TeV.

References

[1] P.M. Stevenson: Phys. Rev. D 30 (1984) 1712

[2] P.M. Stevenson: Phys. Rev. D 32 (1985) 1389

[3] P.M. Stevenson, R. Tarrach: Phys. Lett. B 176 (1986) 436

[4] I. Yotsuyanagi Z. Phys. C - Particles and Fields 35 (1987) 453

[5] J. Marion and S. Thorton *Classical Dynamics of particles and systems*, 4^{th} Edition. Saunders College Publishing, Fort Worth, 1995

[6] J. Townsend *A Modern Approach to Quantum Mechanics* McGraw Hill, New York, 1992

[7] T. Barnes and G. I. Ghandour, Phys. Rev. D 22 (1980) 924

[8] S. J. Chang, Phys. Rev. D 12, (1975) 1071;13 (1976) 2778; Phys. Rep. 23C, (1975) 301

[9] T. L. Curtright and C. B. Thorn, J. Math. Phys. 25 (1984) 541

[10] K. Symanzik, Commun. Math. Phys. 16, (1970) 48

[11] M. Consoli and P. M. Stevenson, Z. Phys. C. **63** (1994) 427.

[12] S. Armstrong. Ph.D. Thesis, University of Wisconsin-Madison, 1998